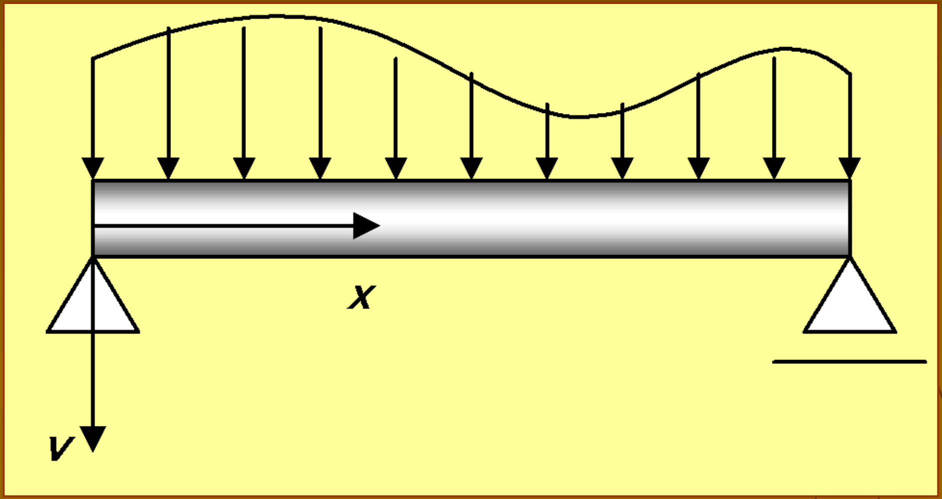
Mechanics of Thin Structure

Mechanics of Thin Structure

What you learned are;

- Introduction for Linear Elasticity
- Stress and Strain with 3D General Expressions
- Plane Stress and Plane Strain
- Principle of Energy
- Principle of Virtual Work
- Calculus of Variations
- Theory of Beams
- Theory of Plates

Example
Pure bending problem

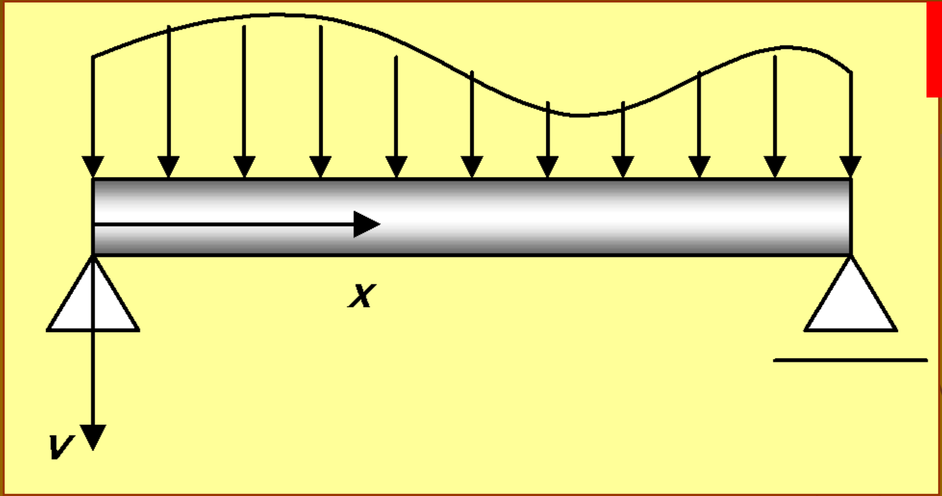


Solve the problem

- **Equilibrium**
- **Compatibility**
- **Energy**

Governing Equation

Example
Pure bending problem



Solve the problem

Equilibrium

Compatibility

Vertical

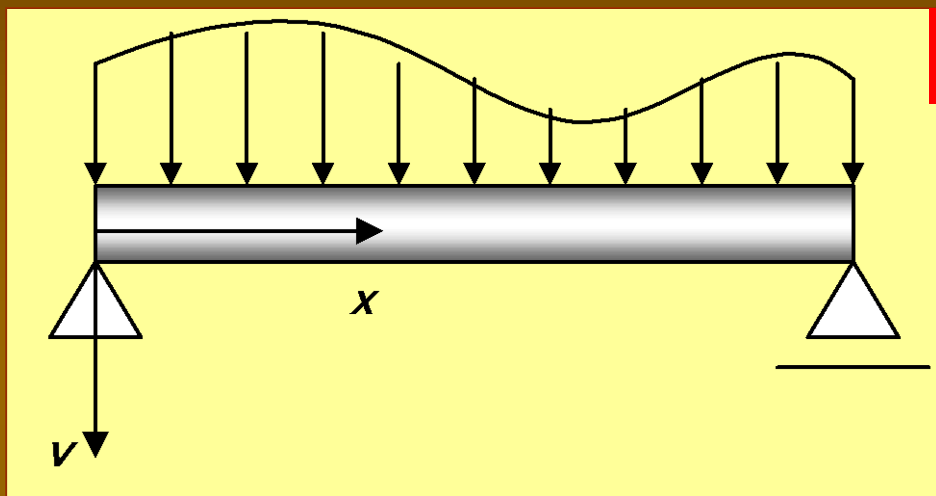
$$-\frac{dQ}{dx} - q(x) = 0$$

Moment

$$-\left(Q + \frac{\partial Q}{\partial x}dx\right)dx - q(x)dx\frac{dx}{2} + \frac{\partial M}{\partial x}dx = 0$$

$$\frac{dM}{dx} = Q \qquad \frac{d^2M}{dx^2} = -q(x)$$

Example
Pure bending problem



Solve the problem

Equilibrium

Compatibility

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0$$

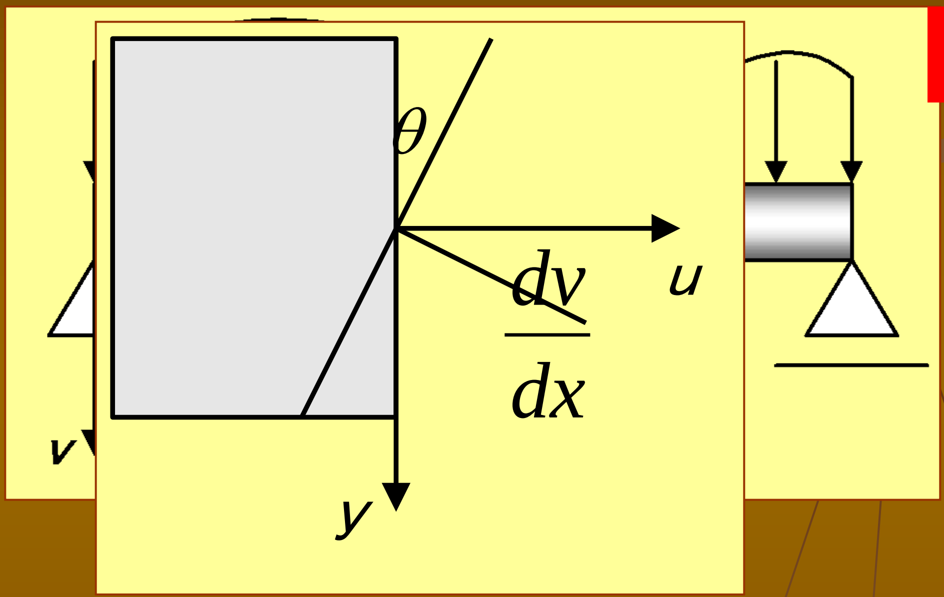
X direction

$$\int_{-h/2}^{h/2} \frac{\partial \sigma_x}{\partial x} z dz + \int_{-h/2}^{h/2} \frac{\partial \tau_{yx}}{\partial y} z dz + \int_{-h/2}^{h/2} \frac{\partial \tau_{zx}}{\partial z} z dz = 0$$

$$\int_{-h/2}^{h/2} \frac{\partial \tau_{zx}}{\partial z} z dz = \tau_{zx} z \Big|_{-h/2}^{h/2} - \int_{-h/2}^{h/2} \tau_{zx} dz$$

$$Q_x = \frac{dM_x}{dx}$$

Example
Pure bending problem



Solve the problem

Equilibrium

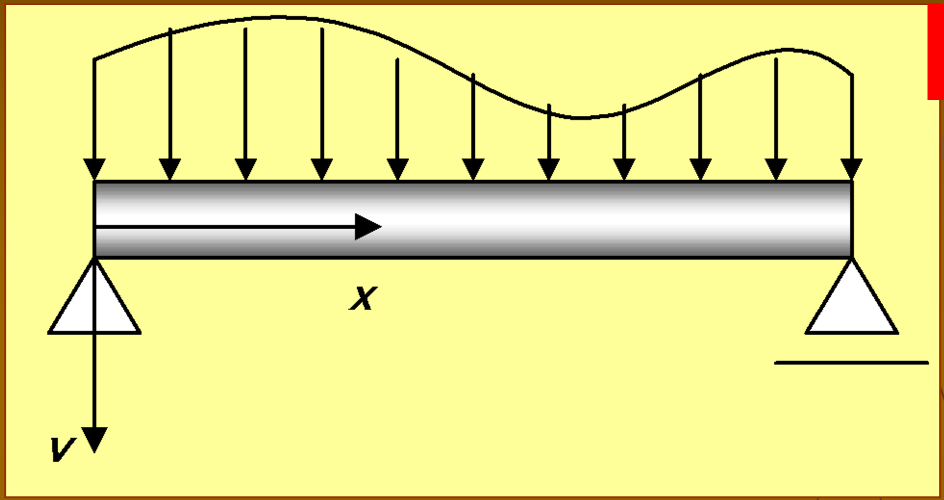
Compatibility

$$u(y) = -y\theta = -y\frac{dv}{dx}$$

$$\varepsilon_x(y) = \frac{du}{dx} = -y\frac{d^2v}{dx^2}$$

$$M_x = \int_A E\varepsilon_x y dA = \int_A E\left(-y\frac{d^2v}{dx^2}\right) y dA = -EI\frac{d^2v}{dx^2}$$

Example
Pure bending problem



Solve the problem

Equilibrium

Compatibility

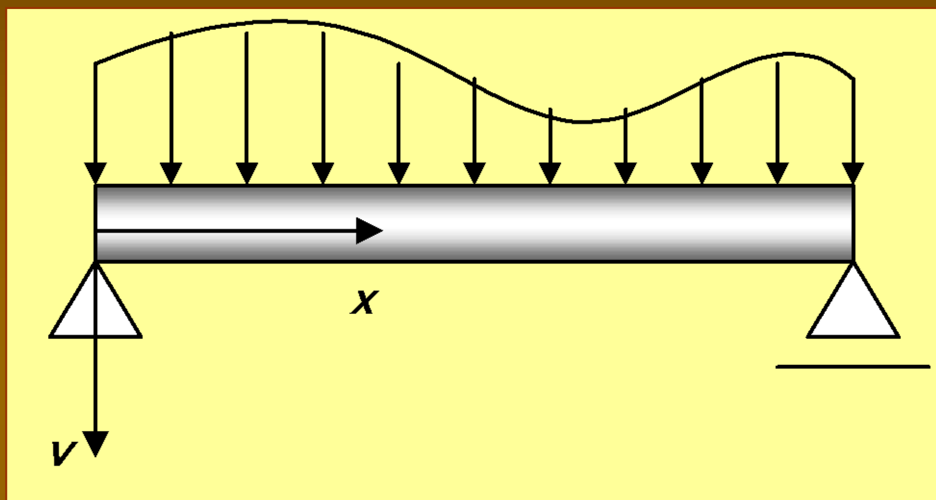
$$\frac{d^2M}{dx^2} = -q(x)$$

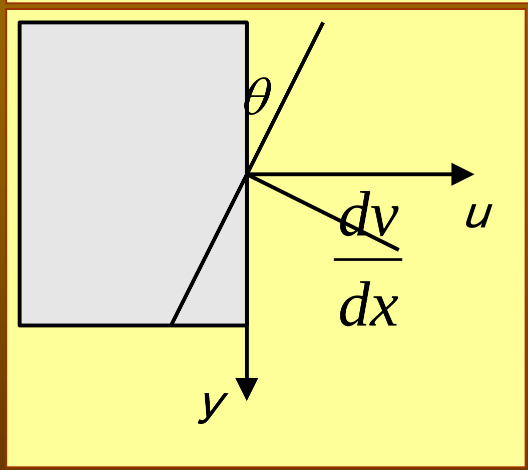
$$M_x = \int_A E\varepsilon_x y dA = \int_A E\left(-y\frac{d^2v}{dx^2}\right) y dA = -EI\frac{d^2v}{dx^2}$$

Governing Equation

$$EI\frac{d^4v}{dx^4} - q(x) = 0$$

Example
Pure bending problem





Compatibility

Energy

$$u(y) = -y\theta = -y\frac{dv}{dx}$$

$$\varepsilon_x(y) = \frac{du}{dx} = -y\frac{d^2v}{dx^2}$$

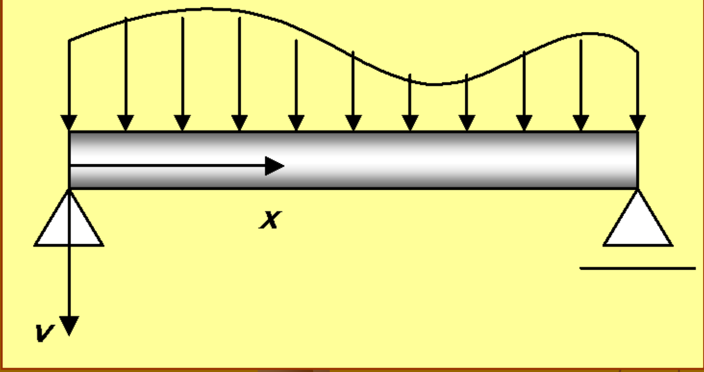
Energy should be Minimum, so that Energy is calcualted such as;

Example
Pure bending problem

Energy for section

$$\int \frac{1}{2}\sigma_x \varepsilon_x b\,dy$$

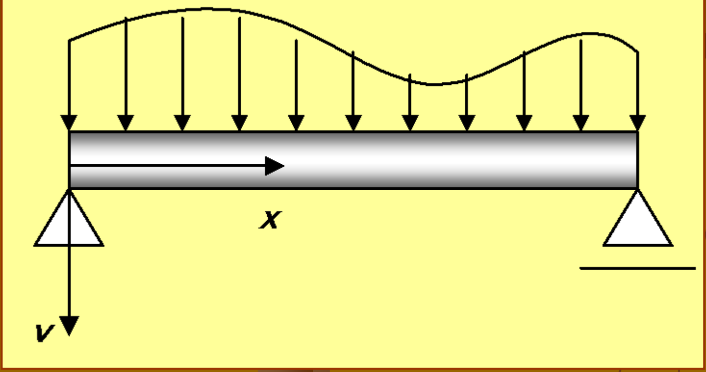
$$\varepsilon_x(y) = \frac{du}{dx} = -y\frac{d^2v}{dx^2}$$



$$U = \int_0^l \left(\int \frac{1}{2}\sigma_x \varepsilon_x b\,dy \right) dx = \int_0^l \left(\int \frac{1}{2}E\varepsilon_x{}^2 b\,dy \right) dx$$

$$= \int_0^l \left(\int \frac{1}{2}Ey^2 \left(\frac{d^2v}{dx^2} \right)^2 b\,dy \right) dx = \frac{EI}{2}\int_0^l \left(\frac{d^2v}{dx^2} \right)^2 dx$$

Example
Pure bending problem



Potential Energy

$$W = \int_0^l q(x) v dx$$

$$\Pi = U - W = \int_0^l \left[\frac{EI}{2} \left(\frac{d^2 v}{dx^2} \right)^2 - q(x) v \right] dx$$

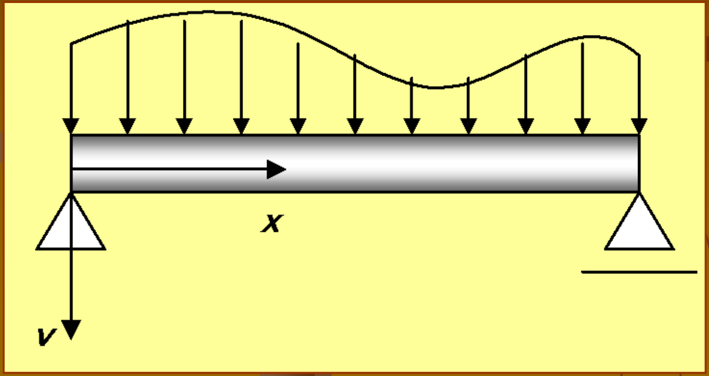
You have the Functional !!

$$\Pi = J[v] = \int_0^l F(x, v, v'') dx$$

Example
Pure bending problem

You have the Functional !!

$$\Pi = J[v] = \int_0^l F(x, v, v'')dx$$



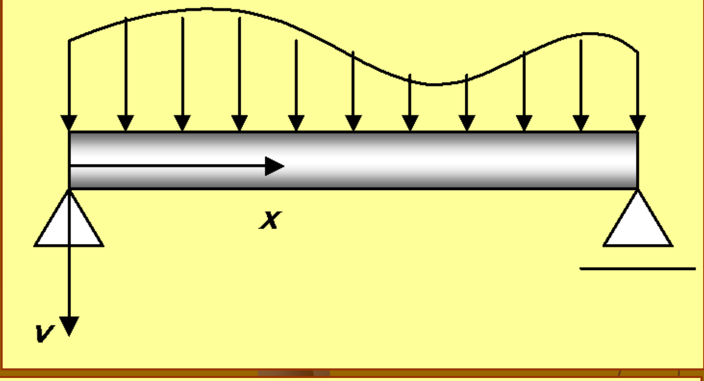
$$\Pi = U - W = \int_0^l \left[\frac{EI}{2} (v'')^2 - q(x)v \right] dx$$

Apply the Euler's Equation

$$\frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial v''}\right) = 0$$

Example
Pure bending problem

$$\frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial v''}\right) = 0$$



$$\frac{\partial F}{\partial v} = -q \qquad \frac{\partial F}{\partial v'} = 0 \qquad \frac{\partial F}{\partial v''} = EIv''$$

$$\Pi = \int_0^l \left[\frac{EI}{2}(v'')^2 - q(x)v \right] dx$$

$$-q + \frac{d^2}{dx^2}(EIv'') = EI\frac{d^4 v}{dx^4} - q = 0$$

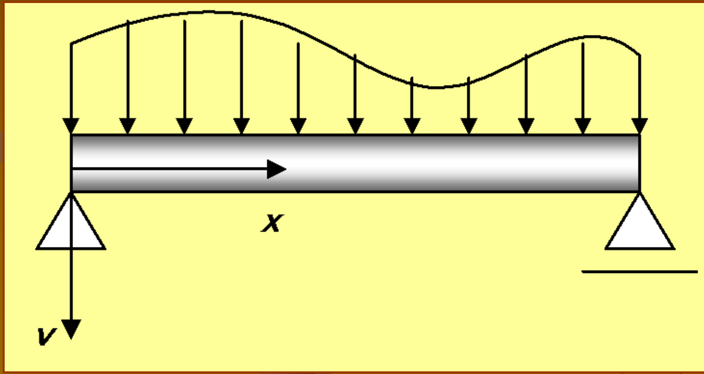
Governing Equation for pure bending beam

from the Euler's Equation

Example
Pure bending problem

Boundary Condition

See eq. (6-31)



$$\frac{\partial F}{\partial v'} - \frac{d}{dx}\left(\frac{\partial F}{\partial v''}\right) = -\frac{d}{dx}\left(EIv''\right) = Q_x$$

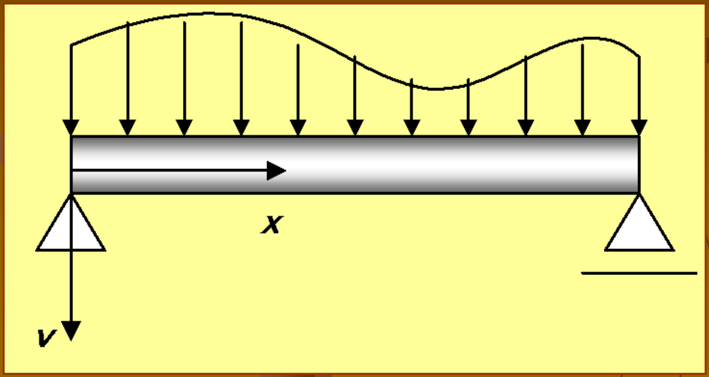
Shearing force δv

$$\frac{\partial F}{\partial v''} = EIv'' = -M_x$$

Bending Moment $\delta v' = \delta \frac{dv}{dx} = \delta\theta$

Example
Pure bending problem

Mechanical Boundary Condition		Geometrical Boundary Condition
Q_x	or	$\delta v' = \delta \dfrac{dv}{dx} = \delta\theta$
M_x	or	δv



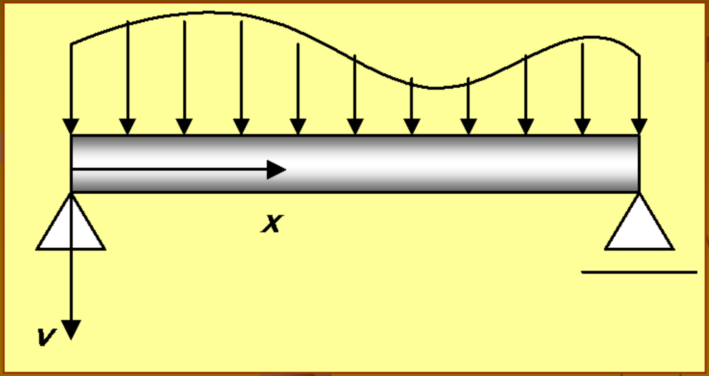
$$\delta J = Q_x \delta v \Big|_{x_1}^{x_2} - M_x \delta\theta \Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left\{ \frac{d^2}{dx^2}\left(EIv''\right) - q \right\} \delta v \, dx$$

Example
Solution 1 : Direct Integration

$$EI\frac{d^4v}{dx^4} - q = 0$$

$$EI\frac{d^3v}{dx^3} = \int_x q(x)dx + C_1$$

$$EI\frac{d^2v}{dx^2} = \int_x\int_x q(x)dxdx + C_1x + C_2$$

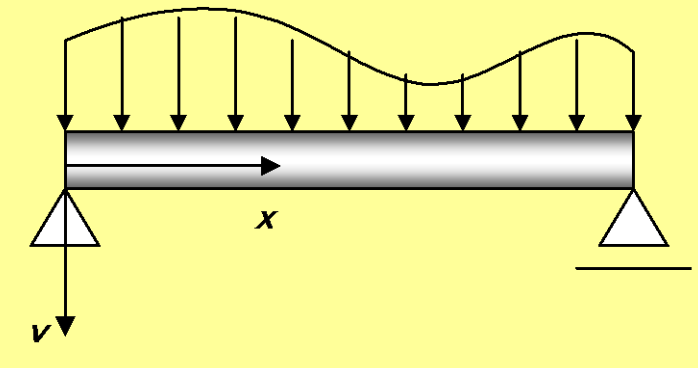


$$\frac{5}{384}\frac{qL^4}{EI} = 0.01302$$

Example
Solution 2 : Virtual Work

$$EI\frac{d^4v}{dx^4} - q = 0$$

Equilibrium



No Energy produced if the Equilibrium is satisfied

$$\int_x \left\{ EI\frac{d^4v}{dx^4} - q \right\} \delta v\,dx = 0$$

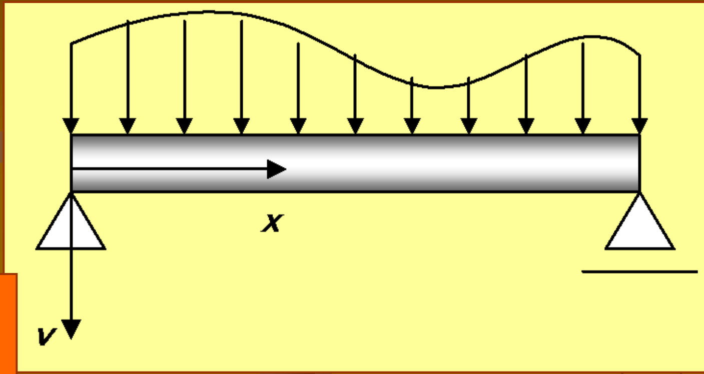
Virtual Displacement

Virtual Work

What is the requirement for the virtual displacement?

Example
Solution 2 : Virtual Work

$$\int_x \left\{ EI\frac{d^4v}{dx^4} - q \right\}\delta v dx = 0$$



$$\tilde{v} = a\sin\left(\frac{\pi}{L}x\right)$$

Assumption

$$EI\left(\frac{\pi}{L}\right)^4 \int_x \left\{ a\sin\left(\frac{\pi}{L}x\right) \right\}\delta v dx - \int_x q\delta v dx = 0$$

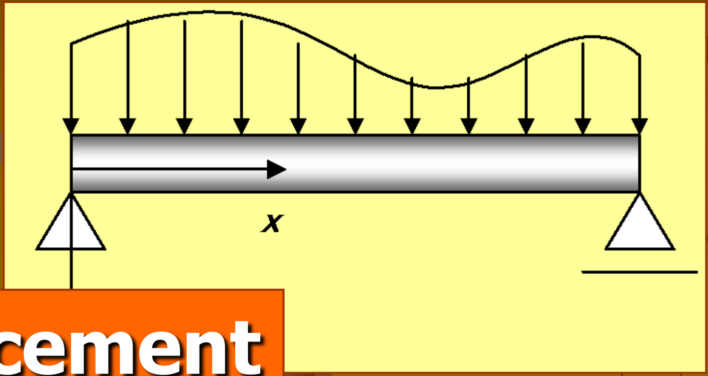
$$\delta v = \sin\left(\frac{\pi}{L}x\right)$$

Virtual Displacement

Example
Solution 2 : Virtual Work

$$\tilde{v} = a\sin\left(\frac{\pi}{L}x\right)$$

Assumption



$$\delta v = \sin\left(\frac{\pi}{L}x\right)$$

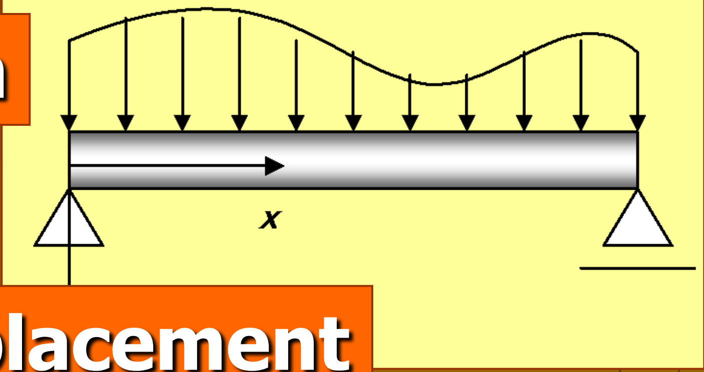
Virtual Displacement

$$EI\left(\frac{\pi}{L}\right)^4 a\int_x \frac{1}{2}\left\{1-\cos\left(2\frac{\pi}{L}x\right)\right\}dx - \int_x q\left\{\sin\left(\frac{\pi}{L}x\right)\right\}dx = 0$$

Example
Solution 2 : Virtual Work

Assumption

$$\tilde{v} = a\sin\left(\frac{\pi}{L}x\right)$$



Virtual Displacement

$$\delta v = \sin\left(\frac{\pi}{L}x\right)$$

$$EI \quad dx - \quad x = 0$$

$$\frac{1}{2}aEI\left(\frac{\pi}{L}\right)^4 L - 2q\left(\frac{L}{\pi}\right) = 0$$

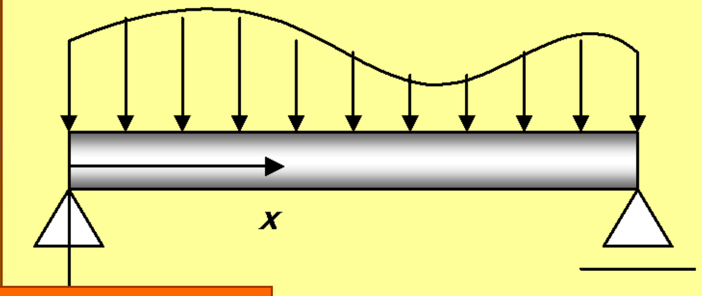
$$a = 4\frac{q}{EI}\left(\frac{L^4}{\pi^5}\right)$$

$$a \approx \frac{4}{306}\frac{qL^4}{EI} = 0.01307\frac{qL^4}{EI}$$

$$\frac{5}{384}\frac{qL^4}{EI} = 0.01302\frac{qL^4}{EI}$$

Example
Solution 2 modified

$$\int_x \left\{ EI \frac{d^4 v}{dx^4} - q \right\} \delta v \, dx = 0$$



$$\tilde{v} = \sum_m a_m \sin\left(\frac{m\pi}{L} x \right)$$

Assumption

$$\sum_m EI \left(\frac{m\pi}{L} \right)^4 \int_x \left\{ a_m \sin\left(\frac{m\pi}{L} x \right) \right\} \delta v \, dx - \int_x q \, \delta v \, dx = 0$$

$$\delta v_n = \sin\left(\frac{n\pi}{L} x \right)$$

Virtual Displacement

Example
Solution 2 modified

$$\sum_m EI\left(\frac{m\pi}{L}\right)^4 \int_x \left\{a_m \sin\left(\frac{m\pi}{L}x\right)\right\}\delta v dx - \int_x q\delta v dx = 0$$

$$\delta v_n = \sin\left(\frac{n\pi}{L}x\right)$$

$$\int_x q\sin\left(\frac{n\pi}{L}x\right)dx$$

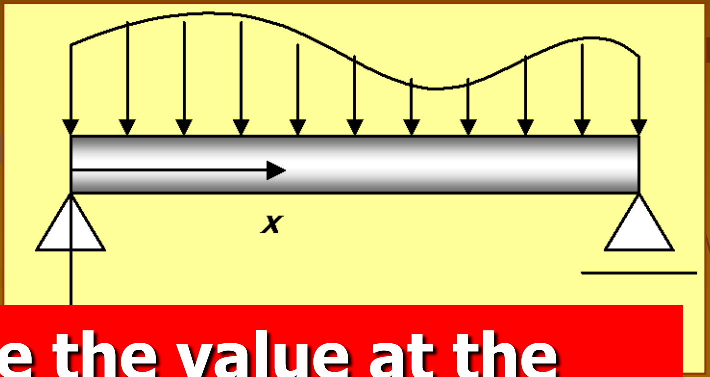
Fourier Transform

$$\sum_m EI\left(\frac{m\pi}{L}\right)^4 \int_x \left\{a_m \sin\left(\frac{m\pi}{L}x\right)\right\}\sin\left(\frac{n\pi}{L}x\right)dx - \int_x q\sin\left(\frac{n\pi}{L}x\right)dx = 0$$

It's same as the solution introduced for Plate Theory

Example
Solution 3 :Point Collocation

$$\int_x \left\{ EI\frac{d^4v}{dx^4} - q \right\} \delta v\, dx = 0$$



Assume you would like to have the value at the center

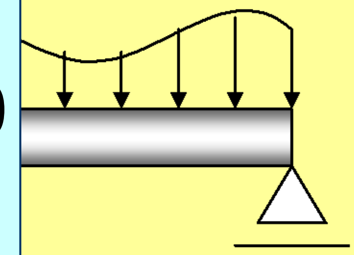
$$\delta v = \delta\left(L/2\right)$$

Dirac Delta Function

$$\delta\left(L/\right. \int_x \left\{ EI\frac{d^4v}{dx^4} - q \right\} \delta dx = \left[EI\frac{d^4v}{dx^4} - q \right]_{x=L/2}$$

Example
Solution 3 :Point Collocation

$$\int_x \left\{ EI\frac{d^4v}{dx^4} - q \right\} \delta dx = \left[EI\frac{d^4v}{dx^4} - q \right]_{x=L/2} = 0$$



$$\tilde{v} = a\sin\left(\frac{\pi}{L}x\right)$$

Assumption

$$\left[EI\frac{d^4v}{dx^4} - q \right] = \left[aEI\left(\frac{\pi}{L}\right)^4 \sin\left(\frac{\pi}{L}x\right) - q \right] = 0$$

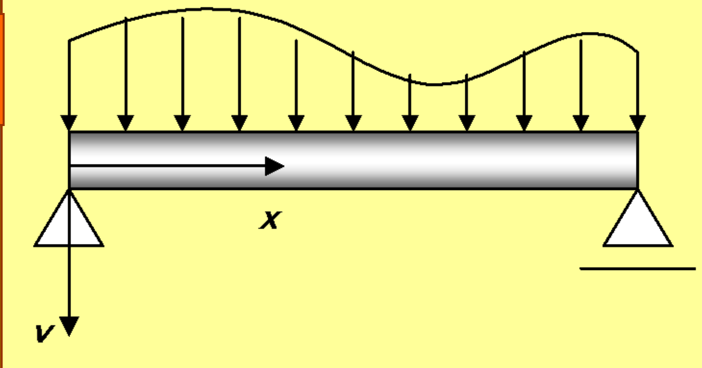
$$a \approx \left(\frac{L}{\pi}\right)^4 \frac{q}{EI} = 0.01027\frac{qL^4}{EI}$$

$$\frac{5}{384}\frac{qL^4}{EI} = 0.01302\frac{qL^4}{EI}$$

Example
Solution 4 :Least Square Method

$$EI\frac{d^4v}{dx^4} - q = 0$$

Equilibrium



Error should be minimum

$$\tilde{v} = a\sin\left(\frac{\pi}{L}x\right)$$

Assumption

$$EI\frac{d^4\tilde{v}}{dx^4} = EIa\left(\frac{\pi}{L}\right)^4 \sin\left(\frac{\pi}{L}x\right)$$

Estimated

Example
Solution 4 :Least Square Method

$$EI\frac{d^4\tilde{v}}{dx^4}-q=EIa\left(\frac{\pi}{L}\right)^4\sin\left(\frac{\pi}{L}x\right)-q$$

Error

$$\Pi=\int_x\left\{EIa\left(\frac{\pi}{L}\right)^4\sin\left(\frac{\pi}{L}x\right)-q\right\}^2dx$$

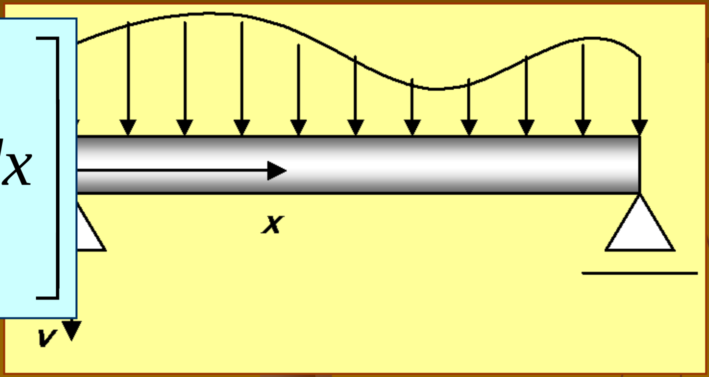
Squared Error

In order to expect the Error minimized, an appropriate value for a must be

$$\frac{\partial\Pi}{\partial a}=\frac{\partial}{\partial a}\left[\int_x\left\{EIa\left(\frac{\pi}{L}\right)^4\sin\left(\frac{\pi}{L}x\right)-q\right\}^2dx\right]$$

Example
Solution 4 :Least Square Method

$$\frac{\partial \Pi}{\partial a}=\frac{\partial}{\partial a}\left[\int_x \left\{EIa\left(\frac{\pi}{L}\right)^4 \sin\left(\frac{\pi}{L}x\right)-q\right\}^2 dx\right]$$



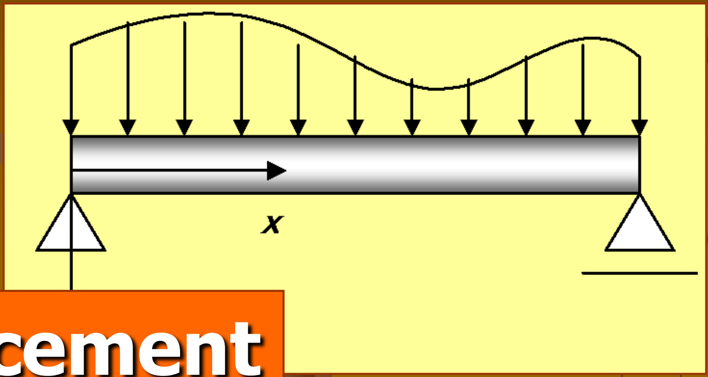
$$\frac{\partial \Pi}{\partial a}=2\int_x \left\{EIa\left(\frac{\pi}{L}\right)^4 \sin\left(\frac{\pi}{L}x\right)-q\right\}EI\left(\frac{\pi}{L}\right)^4 \sin\left(\frac{\pi}{L}x\right)dx=0$$

$$\frac{\partial \Pi}{\partial a}=\int_x \left\{EIa\left(\frac{\pi}{L}\right)^4 \sin\left(\frac{\pi}{L}x\right)-q\right\}\sin\left(\frac{\pi}{L}x\right)dx=0$$

Example
Solution 2 : Virtual Work

$$\tilde{v} = a\sin\left(\frac{\pi}{L}x\right)$$

Assumption

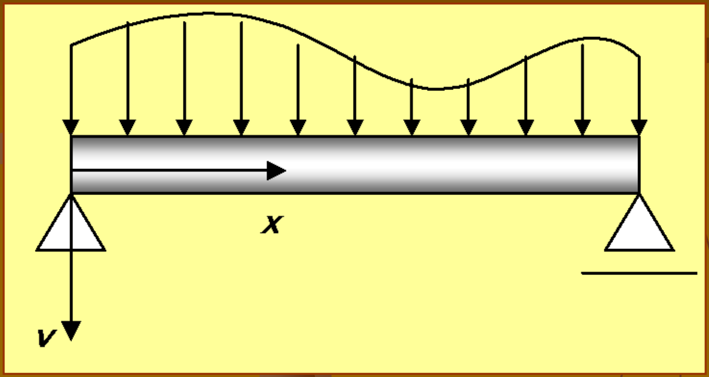


$$\delta v = \sin\left(\frac{\pi}{L}x\right)$$

Virtual Displacement

$$EI\left(\frac{\pi}{L}\right)^4 \int_x \left\{a\sin\left(\frac{\pi}{L}x\right)\right\}\delta v dx - \int_x q\,\delta v dx = 0$$

Example
Solution 4 :Least Square Method

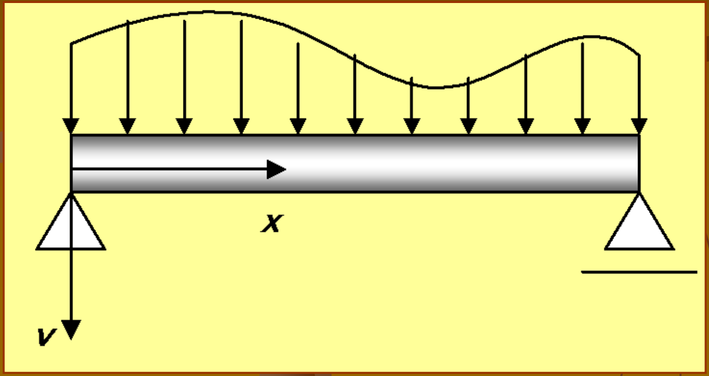


$$\frac{\partial \Pi}{\partial a} = \int_x \left\{ EIa\left(\frac{\pi}{L}\right)^4 \sin\left(\frac{\pi}{L}x\right) - q \right\} \sin\left(\frac{\pi}{L}x\right) dx = 0$$

$$EI\left(\frac{\pi}{L}\right)^4 a \int_x \left\{ \sin\left(\frac{\pi}{L}x\right) \right\}^2 dx - \int_x q\left\{ \sin\left(\frac{\pi}{L}x\right) \right\} dx = 0$$

Example
Solution 5 :Galerkin Method 1

$$\int_x \left\{ EI \frac{d^4 v}{dx^4} - q \right\} \delta v \, dx = 0$$



Starting Equation is Same

$$\tilde{v} = a \sin\left(\frac{\pi}{L} x \right)$$

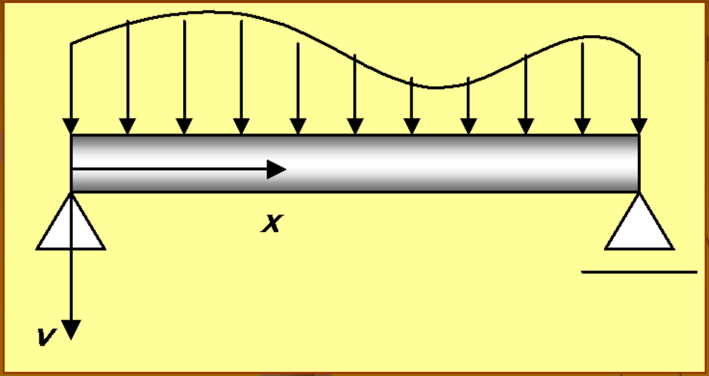
Assumption

$$\delta v = \sin\left(\frac{\pi}{L} x \right)$$

Weighting Function

Example
Solution 5 :Galerkin Method 2

$$\int_x \left\{ EI\frac{d^4 v}{dx^4} - q \right\} \delta v dx = 0$$



Starting Equation is Same

$$\int_x \left\{ EI\frac{d^4 v}{dx^4} - q \right\} \delta v dx = EI\frac{d^3 v}{dx^3}\delta v \Big|_0^L - \int_x EI\frac{d^3 v}{dx^3}\frac{\partial \delta v}{\partial x}dx - \int_x q\delta v dx$$

$$= EI\frac{d^3 v}{dx^3}\delta v \Big|_0^L - EI\frac{d^2 v}{dx^2}\frac{\partial^2 \delta v}{\partial x^2}\Big|_0^L + \int_x EI\frac{d^2 v}{dx^2}\frac{\partial^2 \delta v}{\partial x^2}dx - \int_x q\delta v dx$$

Example
Solution 5 :Galerkin Method 2

$$= EI\frac{d^3v}{dx^3}\delta v\Big|_0^L - EI\frac{d^2v}{dx^2}\frac{\partial^2\delta v}{\partial x^2}\Big|_0^L + \int_x EI\frac{d^2v}{dx^2}\frac{\partial^2\delta v}{\partial x^2}dx - \int_x q\delta v dx$$

$$= \int_x EI\frac{d^2v}{dx^2}\frac{\partial^2\delta v}{\partial x^2}dx - \int_x q\delta v dx = 0$$

$$\tilde{v} = ax^3 + bx^2 + cx + d$$

$$\tilde{v} = ax^3 + bx^2 + cx$$

$$\tilde{v} = aL^3 + bL^2 + cL = 0$$

$$c = -aL^2 - bL$$

$$\tilde{v} = ax\left(x^2 - L^2\right) + bx\left(x - L\right)$$

Example
Solution 5 :Galerkin Method 2

$$\int_x EI \frac{d^2 v}{dx^2}\frac{\partial^2 \delta v}{\partial x^2}dx - \int_x q\delta v dx = 0$$

$$\tilde{v} = ax\left(x^2 - L^2\right) + bx\left(x - L\right)$$

$$\frac{d\tilde{v}}{dx} = 3ax^2 + 2bx + c$$

$$\frac{d^2 v}{dx^2} = 6ax + 2b$$

$$\delta v_1 = x\left(x^2 - L^2\right)$$

$$\delta v_2 = x\left(x - L\right)$$

$$\frac{d^2 \delta v_1}{dx^2} = 6x$$

$$\frac{d^2 \delta v_2}{dx^2} = 2$$

Example Solution 5 :Galerkin Method 2

$$\int_x EI \frac{d^2 v}{dx^2} \frac{\partial^2 \delta v}{\partial x^2} dx - \int_x q \delta v dx = 0$$

$$\frac{d^2 \delta v_1}{dx^2} = 6x$$

$$\frac{d^2 v}{dx^2} = 6ax + 2b$$

$$\delta v_1 = x\left(x^2 - L^2\right)$$

$$\delta v_2 = x\left(x - L\right)$$

$$\frac{d^2 \delta v_2}{dx^2} = 2$$

$$\int_x \{EI(6ax + 2b)6x\} dx - \int_x \{qx(x^2 - L^2)\} dx = 0$$

$$\int_x \{EI(6ax + 2b)2\} dx - \int_x q\{x(x - L)\} dx = 0$$

Example
Solution 5 :Galerkin Method 2

$$EI\left\{12aL^3+6bL^2\right\}=\frac{L^4}{4}q$$

$$a=0$$

$$EI\left\{6aL^2+4bL\right\}=\frac{L^3}{6}q$$

$$b=\frac{qL^2}{24EI}$$

$$\tilde{v}=\frac{qL^2}{24EI}x\left(x-L\right)$$

$$v=\frac{qL^4}{96EI}=0.0104166\frac{qL^4}{EI}$$

$$\frac{5}{384}\frac{qL^4}{EI}=0.01302\frac{qL^4}{EI}$$

Example Solution 5 :Galerkin Method 3

$$\tilde{v} = ax^3 + bx^2 + cx + d$$

$$\frac{d\tilde{v}}{dx} = 3ax^2 + 2bx + c$$

$$\tilde{v} = d = \tilde{v}_1$$

$$\tilde{v} = aL^3 + bL^2 + cL = 0 = \tilde{v}_2$$

$$c = -aL^2 - bL = \theta_1$$

$$3aL^2 + 2bL + c = 2aL^2 + bL = \theta_2$$

$$a = \frac{\theta_1 + \theta_2}{L^2}$$

$$b = \frac{-\left(2\theta_1 + \theta_2\right)}{L}$$

$$\tilde{v} = \left\{\frac{x^3}{L^2} - 2\frac{x^2}{L} + x\right\}\theta_1 + \left\{\frac{x^3}{L^2} - \frac{x^2}{L}\right\}\theta_2$$

Example
Solution 5 :Galerkin Method 3

$$\tilde{v} = \left\{ \frac{x^3}{L^2} - 2\frac{x^2}{L} + x \right\}\theta_1 + \left\{ \frac{x^3}{L^2} - \frac{x^2}{L} \right\}\theta_2$$

$$\frac{d^2\tilde{v}}{dx^2} = \left\{ 6\frac{x}{L^2} - 4\frac{1}{L} \right\}\theta_1 + \left\{ 6\frac{x}{L^2} - 2\frac{1}{L} \right\}\theta_2$$

$$\int_x EI \frac{d^2 v}{dx^2}\frac{\partial^2 \delta v}{\partial x^2}dx - \int_x q\delta v dx = 0$$

Example
Solution 5 :Galerkin Method 3

$$EI\left\{\frac{4}{L}\theta_1+\frac{2}{L}\theta_2\right\}-\left\{\frac{L^2}{12}q\right\}=0$$

$$EI\left\{\frac{2}{L}\theta_1+\frac{4}{L}\theta_2\right\}-\left\{-\frac{L^2}{12}q\right\}=0$$

$$EI\,\frac{6}{L}\theta_1=\left\{\frac{3L^2}{12}q\right\}$$

$$\theta_1=\frac{qL^3}{24EI}$$

$$\theta_2=-\frac{qL^3}{24EI}$$

Example
Solution 5 :Galerkin Method 3

$$\tilde{v} = \left\{ \frac{x^3}{L^2} - 2\frac{x^2}{L} + x \right\}\theta_1 + \left\{ \frac{x^3}{L^2} - \frac{x^2}{L} \right\}\theta_2$$

$$\theta_1 = \frac{qL^3}{24EI}$$

$$\theta_2 = -\frac{qL^3}{24EI}$$

$$\tilde{v} = \left\{ \frac{L}{8} - \frac{L}{2} + \frac{L}{2} \right\}\theta_1 + \left\{ \frac{L}{8} - \frac{L}{4} \right\}\theta_2 = \frac{qL^4}{96EI}$$

$$v = \frac{qL^4}{96EI} = 0.0104166\frac{qL^4}{EI}$$

$$\frac{5}{384}\frac{qL^4}{EI} = 0.01302\frac{qL^4}{EI}$$

Mechanics of Thin Structure

What you learned are;

- Introduction for Linear Elasticity
- Stress and Strain with 3D General Expressions
- Plane Stress and Plane Strain
- Principle of Energy
- Principle of Virtual Work
- Calculus of Variations
- Theory of Beams
- Theory of Plates